

Modulational instability and nonlinear evolution of two-dimensional electrostatic wave packets in ultra-relativistic degenerate dense plasmas

Amar Prasad Misra^{1, a)} and Padma Kant Shukla^{1, 2, b)}

¹⁾*Department of Physics, Umeå University, SE-901 87 Umeå, Sweden*

²⁾*RUB International Chair, International Centre for Advanced Studies in Physical Sciences, Faculty of Physics & Astronomy, Ruhr University Bochum, D-44780 Bochum, Germany*

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We consider the nonlinear propagation of electrostatic wave packets in an ultra-relativistic (UR) degenerate dense electron-ion plasma, whose dynamics is governed by the nonlocal two-dimensional nonlinear Schrödinger-like equations. The coupled set of equations are then used to study the modulational instability (MI) of a uniform wave train to an infinitesimal perturbation of multi-dimensional form. The condition for the MI is obtained, and it is shown that the nondimensional parameter, $\beta \propto \lambda_C n_0^{1/3}$ (where λ_C is the reduced Compton wavelength and n_0 is the particle number density), associated with the UR pressure of degenerate electrons, shifts the stable (unstable) regions at $n_0 \sim 10^{30} \text{ cm}^{-3}$ to unstable (stable) ones at higher densities, i.e. $n_0 \gtrsim 7 \times 10^{33}$. It is also found that the higher the values of n_0 , the lower is the growth rate of MI with cut-offs at lower wave numbers of modulation. Furthermore, the dynamical evolution of the wave packets is studied numerically. We show that either they disperse away or they blowup in a finite time, when the wave action is below or above the threshold. The results could be useful for understanding the properties of modulated wave packets and their multi-dimensional evolution in UR degenerate dense plasmas, such as those in the interior of white dwarfs and/or pre-Supernova stars.

^{a)}Electronic mail: apmisra@visva-bharati.ac.in; apmisra@gmail.com; On leave from Department of Mathematics, Siksha Bhavana, Visva-Bharati University, Santiniketan-731 235, India.

^{b)}Electronic mail: ps@tp4.rub.de; profshukla@yahoo.com

I. INTRODUCTION

As is known, compact astrophysical objects, e.g. massive white dwarfs and/or the core of pre-Supernova stars, are supported by the pressure of degenerate electrons, i.e., in their interiors the particle number density is extremely high^{1,2}. So, the Fermi energy can be much larger than the thermal energy (e.g. typical electron Fermi energy, $E_F \sim 1$ MeV corresponds to a Fermi temperature $T_F \sim 10^{10}$ K). Thus, the thermal pressure of electrons may be negligible compared to the Fermi pressure. Such degenerate electrons may, however, be either nonrelativistic, somewhat relativistic or ultra-relativistic. In the latter case, the speed of electrons can approach the speed of light in vacuum (c), and the equation of state for electrons can then be written as²⁻⁴ $P_e = (3/\pi)^{1/3} (\hbar c/8) n_e^{4/3}$ whenever $E_F \gg m_e c^2$. Here, \hbar is the reduced Planck's constant, n_e is the electron number density and m_e is the electron mass. Such ultra-relativistic degenerate (URD) electrons are, indeed, ubiquitous in many other astrophysical environments including neutron stars and magnetars^{1,2,5}.

Now, in dealing with those compact objects like white dwarfs, their interiors may be considered as a plasma system (e.g., carbon-oxygen white dwarf in a thermonuclear Supernova explosion) consisting of positively charged ions (nuclei) providing almost all the mass (inertia) and none of the pressure, as well as electrons providing all the pressure (restoring force), but none of the mass (inertialess). So, interiors of such compact objects provide us a cosmic laboratory for studying the properties of such plasmas as well as nonlinear collective oscillations under extreme conditions, i.e. at the relativistically degenerate dense states with higher densities⁵ ($10^6 - 10^9$ g/cm³). Recent investigations along these lines indicate that such URD dense plasmas can support the propagation of solitary waves as well as double layers at different length scales of excitation⁶.

On the other hand, the nonlinear propagation of wave packets in plasmas is generically subject to their amplitude modulation due to the carrier wave self-interaction, i.e., a slow variation of the wave envelope due to nonlinearities (see, e.g.⁷⁻¹⁰). Under certain conditions, the system's evolution shows a modulational instability (MI), leading to the formation of envelope solitons through the localization of wave energy. Such solitons, governed by a (1+1)-dimensional nonlinear Schrödinger equation (NLSE), are the result of a balance between the nonlinearity (self-focusing) and the wave group dispersion. When such a balance can be maintained dynamically, the solitons may exist even under strong perturbations. On the contrary, the multi-dimensional [(2+1) or (3+1)-dimensional] NLSE with cubic and/or quadratic (nonlocal) nonlinearities may no longer be integrable. In this case, the self-focusing effect can dominate over the dispersion (beyond a threshold intensity) leading to the formation of wave collapse or blowup of the wave amplitude^{11,12}.

The important issue for the multi-dimensional NLSE may be that for a wide range of initial conditions, the system often exhibits collapse in which a singularity of the wave field is formed in a finite time, instead of a stable mode propagation. A collapsing wave packet thus self-focuses in shorter scales and with higher amplitudes until other physical effects intervene to arrest it. Such collapse phenomenon plays an important role as an effective mechanism for the energy localization in various branches of physics^{11,12}. In this context, it could be a central problem of finding a proper initial condition which leads to wave collapse. This is, one of our goals of the present work in the description of Davey-Stewartson II (DS II)-like equations¹³ for the wave packets. Several authors have investigated such collapse theory, e.g. in the context of hydrodynamics¹⁴, in nonlinear optics¹⁵, in Bose-Einstein condensates¹⁶ as well as in plasma physics¹². However, to our knowledge, there are few theoretical studies for the MI (see, e.g.¹⁷⁻¹⁹) and multi-dimensional evolution of electrostatic wave packets (see, e.g.¹⁹) in plasmas, which are described by the DS II-like equations. The latter generalize the (1+1)-dimensional NLSE with a nonlocal (quadratic) nonlinear term associated with the static field in the plasma.

Furthermore, since quadratic nonlinearities are known to be collapse-free, multi-dimensional NLSE, where the cubic and quadratic (nonlocal) nonlinearities compete, can also support coherent structures, i.e. dromion-like solutions (unlike solitons the dromions can have inelastic collisions and can transfer mass or energy) which may either decay due to the dispersion to be enhanced by the static field or exhibit blowup due to nonlinearity, in a finite time^{20,21}. However, their applications in physical systems, especially in plasmas are not yet fully understood or less developed till now.

In this article, our purpose is to consider the propagation of two-dimensional (2D) electrostatic wave packets (EWPs) in a URD dense plasma. We provide a general criterion for the MI of a plane wave packet as well as the instability growth rate. We show that as one approaches the higher density regimes, the stability of the wave increases with lower growth rates at lower wave numbers of modulation. Furthermore, the 2D evolution of the nonlocal NLSEs exhibit dromion-like solutions which either decay by the wave dispersion or blowup due to wave nonlinearity in a finite interval of time. The latter are in qualitative agreement with the results^{20,21} already found in DS II equations for water waves with finite depth¹³.

II. BASIC EQUATIONS AND DERIVATION OF THE EVOLUTION EQUATIONS

Let us consider the propagation of EWPs in a 2D dense plasma composed of inertialess URD ultra-cold electrons and inertial ultra-cold ions. Any speed involved in the plasma flow is assumed to be much lower than the ion-acoustic speed. At equilibrium, both species have equal number density, say n_0 . Assuming further that the plasma is collisionless and unmagnetized, the basic normalized equations then read⁶

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \phi, \quad (2)$$

$$0 = \nabla \phi - \frac{3\beta}{4n_e} \nabla n_e^{4/3}, \quad (3)$$

$$\nabla^2 \phi = n_e - n_i, \quad (4)$$

where $\nabla \equiv (\partial/\partial x, \partial/\partial y)$, n_i is the ion number density normalized by n_0 , $\mathbf{v} \equiv (v_x, v_y)$ is the ion velocity normalized by $C_i = \sqrt{Z_i m_e c^2 / m_i}$ with Z_i denoting the ion charge state, m_i is the ion mass and c is the speed of light in vacuum. Also, ϕ is the electrostatic wave potential normalized by $Z_i m_e c^2 / e$ with e denoting the elementary charge. The space and time variables are respectively normalized by the screening length $\lambda_s = \sqrt{m_e c^2 / 4\pi n_0 e^2}$ and the ion plasma period, $\omega_{pi}^{-1} = \sqrt{m_i / 4\pi n_0 Z_i^2 e^2}$. Moreover, in Eq. (3), $\beta = \lambda_C \sqrt[3]{n_0 / 72\pi}$ where $\lambda_C \equiv \hbar / mc$ is the reduced Compton wavelength, and we have used the same equation of state for P_e as described in the previous section relevant for URD electrons in dense plasmas. Furthermore, the nondimensional parameter, $\beta \gtrsim 1$ for $n_0 \gtrsim n_c \approx 3.94 \times 10^{33} \text{ cm}^{-3}$ and $\beta = 1$ for $n_0 = n_c$, i.e. higher values of β represent the higher-density regimes.

Next, we consider the time evolution of a wave packet of electrostatic perturbations which occur along the x -axis. The initial wave packets of perturbation are basically modulated by the nonlinear carrier wave self-interactions. Then if one observes the wave packets from a coordinate frame moving with a group speed v_g , to be obtained from the linear dispersion relation (as given below) of Eqs. (1)-(4), then the time variation of the wave packets looks slow, and so the space and the time variables can be stretched as

$$\xi = \epsilon(x - v_g t), \eta = \epsilon y, \tau = \epsilon^2 t, \quad (5)$$

where ϵ is a small parameter representing the strength of the wave amplitude. We are interested in the modulation of a plane wave as the carrier wave with wave number and frequency k and ω respectively, then following the standard reductive perturbation technique (RPT) (see, e.g.¹⁹), the dynamical variables can be expanded as

$$n_{e,i} = 1 + \sum_{n=1}^{\infty} \epsilon^n \sum_{l=-\infty}^{\infty} n_{(e,i)l}^{(n)}(\xi, \eta, \tau) \exp[i(kx - \omega t)l], \quad (6)$$

$$v_{x,y} = \sum_{n=1}^{\infty} \epsilon^n \sum_{l=-\infty}^{\infty} v_{(x,y)l}^{(n)}(\xi, \eta, \tau) \exp[i(kx - \omega t)l], \quad (7)$$

$$\phi = \sum_{n=1}^{\infty} \epsilon^n \sum_{l=-\infty}^{\infty} \phi_l^{(n)}(\xi, \eta, \tau) \exp[i(kx - \omega t)l], \quad (8)$$

where $n_{(e,i)l}^{(n)}$, $v_{(x,y)l}^{(n)}$ and $\phi_l^{(n)}$ should satisfy $S_{-l}^{(n)} = S_l^{(n)*}$ because of the reality condition for the physical variables. Here, the asterisk denotes the complex conjugate of the corresponding quantity. Note that the group speed, v_g is now normalized by C_i (smaller than the ion-sound speed), the wave frequency ω and the wave number k are normalized by ω_{pi} and λ_s respectively.

We now substitute the expressions (6)-(8) into the basic Eqs. (1)-(4) and equate the terms in different powers of ϵ . We shall, however, omit the detail calculations, since the procedure is quite standard and follows the usual RPT. In the lowest order of ϵ , we obtain for $n = 1, l = 1$ the linear dispersion relation [since we are considering the modulation of a plane wave, $n_{(e,i)l}^{(1)}$, $v_{(x,y)l}^{(1)}$ and $\phi_l^{(1)}$ are all set to zero except for $l = \pm 1$]

$$\omega^2 = \frac{\beta k^2}{1 + \beta k^2}. \quad (9)$$

We find that the dispersion equation (9) has the similar form with that derived by Kako *et al*²² for classical plasmas comprising cold ions and isothermal electrons whenever one replaces β by unity. However, one should note that the

case $\beta = 1$, which represents the higher density regimes where $n_0 = n_c$, is not applicable for classical isothermal plasmas. Equation (9) shows that the wave always propagates with a frequency below the ion-plasma frequency regardless of the values of β and k . In the short-wavelength limit, the frequency of the wave approaches unity (the upper limit of the normalized ω), whereas in the long-wavelength limit, it approaches a zero value. Furthermore, the limit $\beta k^2 \ll 1$ is not admissible because otherwise, ω will be small to provide weak dispersion, and the soliton formation (Korteweg-de Vries soliton) of the carrier wave will be a lower-order process than the MI of the envelope to be studied here. Moreover, since k is normalized by the inverse of the screening length λ_s , the values of $k > 2\pi$ are also inadmissible, otherwise the wavelength would become smaller than the screening length. As a result, the plasma collective behaviors might disappear.

Now, proceeding in the same way as of Refs.^{17–19}, i.e., considering the second harmonic modes obtained in terms of $\phi_1^{(2)}$, $\partial\phi_1^{(1)}/\partial\xi$, $\partial\phi_1^{(1)}/\partial\eta$ for $n = 2, l = 1$ and $[\phi_1^{(1)}]^2$ for $n = 2, l = 2$ as well as the zeroth-harmonic modes appearing due to the nonlinear self-interaction of the modulated carrier waves, and finally considering the equations for $n = 3, l = 1$, we obtain the following 2D nonlocal NLSEs for the propagation of modulated wave packets in URD dense plasmas

$$i\frac{\partial\phi}{\partial\tau} + P_1\frac{\partial^2\phi}{\partial\xi^2} + P_2\frac{\partial^2\phi}{\partial\eta^2} + Q_1|\phi|^2\phi + Q_2\psi\phi = 0, \quad (10)$$

$$R\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = S\frac{\partial^2|\phi|^2}{\partial\xi^2}, \quad (11)$$

where $\phi \equiv \phi_1^{(1)}$, $\psi \equiv \int \partial_\xi v_{y0}^{(2)} \partial\eta$ and the coefficients written only in terms of k and the parameter β are given by

$$P_1 = -\frac{3k\beta^{3/2}}{2(1+\beta k^2)^{5/2}}, \quad P_2 = \frac{\sqrt{\beta}}{2k(1+\beta k^2)^{3/2}}, \quad (12)$$

$$Q_1 = \frac{k\sqrt{1+\beta k^2}}{9\sqrt{\beta}} \left[-\frac{3}{2} + \frac{7\beta^4}{3(1+\beta k^2)^2} \right] \left(2 + 3k^2 + \frac{2}{\beta^2 k^2} \right) \\ + \frac{k(1+\beta k^2)^{3/2}}{\beta^{3/2}} \left[1 + \frac{\beta^3(5-2/\beta^3)}{3(1+\beta k^2)^3} \right] - \frac{k(-2+3\beta+3\beta^2 k^2)}{9\beta^{5/2}(1+\beta k^2)^{3/2}}, \quad (13)$$

$$Q_2 = \frac{k(5-3\beta k^2)}{6\sqrt{\beta}(1+\beta k^2)^{3/2}}, \quad R = 1 - \frac{1}{(1+\beta k^2)^3}, \quad (14)$$

$$S = -\frac{1+\beta k^2}{\beta^{3/2}} \left[3 + \beta k^2 - \frac{2}{3(1+\beta k^2)^3} \right]. \quad (15)$$

We note that since, $P_1 < 0$, $P_2 > 0$ and $R > 0$, the NLSEs (10) and (11) are in the form of second type (hyperbolic-elliptic) Davey-Stewartson, i.e. DS II-like equations. Furthermore, S is always negative, $Q_2 \geq 0$ for $\beta k^2 \leq 5/3$, $Q_2 = 0$ for $\beta k^2 = 5/3$ and Q_1 can be either positive or negative depending on the values of both k and β . In the case $Q_2 = 0$ for which Eqs. (10) and (11) are reduced to 2D NLSE, the modulated wave can be shown to be stable. This will be clear from the next section dealing with the MI. We will see that for values of $\beta > 1$ (which corresponds to higher density regimes) and small wave numbers, i.e. $k < 1$ fulfilling $\beta k^2 \lesssim 3$, the wave tends to become stable.

Thus, we have obtained a new set of nonlocal NLSEs, which describe the slow modulation of EWPs in 2D URD dense plasmas. The coefficients P_1, P_2 appear due to the wave group dispersion and the 2D evolution of the EWPs. One of the nonlinear coefficients, Q_1 (cubic) is due to the carrier wave self-interaction originating from the zeroth harmonic modes (or slow modes) and the other nonlocal coefficient (quadratic) Q_2 comes from the coupling between the dynamical field associated with the first harmonic (with a ‘cascaded’ effect from the second harmonic) and a static field generated due to the mean motion (zeroth harmonic) in plasmas.

III. MODULATIONAL INSTABILITY AND GROWTH RATE

In this section, we consider the modulation of a plane wave solution of Eqs. (10) and (11) for ϕ with a constant amplitude ϕ_0 . The boundary conditions, namely $\phi, \psi \rightarrow 0$ as $\xi, \eta \rightarrow \infty$ used before must now be relaxed, since the wave packet is still not modulated and the solution is not unique. Note that the choice of $\psi = \psi_0$ is immaterial as the stability criterion does not depend on it. Thus, we can represent the plane wave solution as $\phi = \phi_0 \exp[i(k_\xi \xi + k_\eta \eta - \Delta\tau)]$,

and $\psi = \psi_0$ with $\Delta = P_1 k_\xi^2 + P_2 k_\eta^2 - Q_1 \phi_0^2 - Q_2 \psi_0$, where k_ξ, k_η, ψ_0 are all real constants. Next, to study the stability of this solution we modulate the amplitude as $\phi = (\phi_0 + \phi_m) \exp[i(k_\xi \xi + k_\eta \eta - \Delta \tau)]$, $\psi = \psi_0 + \psi_m$, where $\phi_m, \psi_m \propto \text{Re}[\exp i(K_1 \xi + K_2 \eta - \Omega \tau)]$ and K, Ω are respectively the wave number and the wave frequency of modulation. Looking for the nonzero solution of the small amplitude perturbations, we obtain from Eqs. (10) and (11) the following dispersion relation for the modulated wave packet

$$\Omega^2 = (P_1 K_1^2 + P_2 K_2^2)^2 (1 - K_c^2/K^2), \quad (16)$$

where

$$K_c^2 = \frac{2\phi_0^2 [(R + \zeta^2) Q_1 + S Q_2] (1 + \zeta^2)}{(P_1 + P_2 \zeta^2) (R + \zeta^2)}, \quad (17)$$

in which $\zeta \equiv \tan \theta = K_2/K_1$. Equation (16) shows that the MI sets in for a wave number satisfying $K < K_c$ or, for all wavelengths above the threshold, $\lambda_c = 2\pi/K_c$, provided the right-hand side of Eq. (17) is positive, i.e.

$$\frac{(R + \zeta^2) Q_1 + S Q_2}{(P_1 + P_2 \zeta^2) (R + \zeta^2)} > 0. \quad (18)$$

Since $R > 0$, the condition (18) reduces to

$$\Lambda \equiv [(R + \zeta^2) Q_1 + S Q_2] (P_1 + P_2 \zeta^2) > 0. \quad (19)$$

The wave packet is otherwise (i.e., for $K > K_c$) said to be stable under the modulation. The instability growth rate (letting $\Omega = i\Gamma$) can be obtained from Eq. (16) as

$$\Gamma = \frac{K^2 (P_1 + P_2 \zeta^2)}{1 + \zeta^2} \sqrt{\frac{K_c^2}{K^2} - 1}. \quad (20)$$

Clearly, the maximum growth rate, achieved at $K = K_c/\sqrt{2}$, is $\Gamma_{\max} = [(R + \zeta^2) Q_1 + S Q_2] |\phi_0|^2 / (R + \zeta^2)$.

Next, we numerically investigate the condition of MI given by Eqs. (18) or (19) as well as the instability growth rate, Γ given above. The condition of MI not only depends on the carrier wave number k and the density dependent parameter β arising due to the ultra-relativistic pressure of degenerate electrons, but also on the obliqueness parameter θ of the modulational wave number K with the ξ -axis due to 2D perturbation. The stable ($\Lambda < 0$) and unstable ($\Lambda > 0$) regions are thus shown in the $k\theta$ plane in Fig. 1 for different values of β that correspond to different density regimes. We find that the stable and unstable regions are completely divided into two parts at comparatively lower as well as higher density plasmas, i.e. at $n_0 \sim 10^{30} \text{ cm}^{-3}$ [see Fig. 1(a)] and at $n_0 = 5 \times 10^{33}, 7 \times 10^{33}$ and 10^{34} cm^{-3} [see Figs. 1(d)-1(f)]. Separation of such regions are also observed at the intermediate densities [Figs. 1(b) and (c)]. As seen from these figures, there are basically four regions (two for each) for the stable and unstable waves, and for a wide range of values of k and $\tan \theta$. From Figs. 1(a) and 1(e) or 1(f), one observes that some part of the regions in which the wave was stable (unstable) at lower density now shifts to unstable (stable) region at higher values of the same. Obviously, the parameter responsible for this shift is β due to the consideration of ultra-relativistic degenerate electrons.

Again, P_2, R are always positive and $Q_2 > 0$ for $\beta k^2 < 5/3$. Also, $S < 0$ and $P_2 > |P_1|$ for $\beta k^2 > 1/2$. Then, in the regime satisfying $1/2 < \beta k^2 < 5/3$, the instability condition (18) or (19) depends mainly on the sign and magnitude of the coefficient Q_1 . Physically, this implies that the EWPs become stable or unstable in the said regime due to the nonlinear self-interactions originating from the second harmonic modes as well as from the zeroth harmonic modes (or slow modes). In particular, under the horizontal modulation ($K_1 = 0$) the instability condition (18) reduces to $\Lambda = Q_1/P_2$ which can be either positive or negative depending on the sign of Q_1 (since $P_2 > 0$). We also find that for a particle density to vary in $10^{32} - 10^{33} \text{ cm}^{-3}$ and for carrier wave numbers $k < 1$, Q_1 is negative, and then the modulated wave is said to be stable. On the other hand, under the longitudinal modulation, i.e. $K_2 = 0$, we have $\Lambda = (Q_1 R + S Q_2)/P_1$, which may, however, be positive almost everywhere in the above regime, and hence the instability. These results are in qualitative agreement with the classical ones considered before by Nishinari et al¹⁹ in the description of DS II equations for EWPs. However, the general situation is quite different as clear from Fig. 1.

Now, the MI growth rate can be calculated from Eq. (20) for a fixed value of the carrier wave number. Figure 2 shows that higher the density regimes, the lower is the growth rate of instability with cut-offs at lower wave numbers of modulation. It seems that the parameter β plays almost the similar role of dispersion as the quantum diffraction associated with the Bohm potential plays for modulated wave packets in quantum plasmas⁷⁻¹⁰. This is expected as

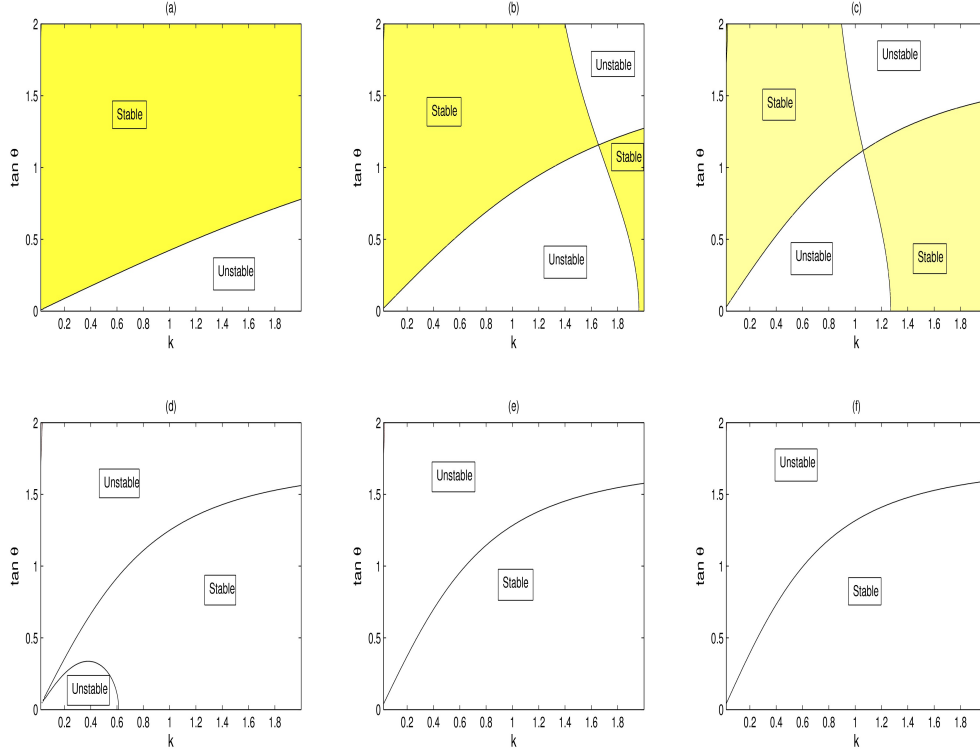


FIG. 1. (Color online) Contour plots of $\Lambda = \text{const.}$ against k and $\tan \theta$. The stable (shaded and/or wherever mentioned) and unstable (white and/or wherever mentioned) regions are shown in the $k\theta$ -plane. The panels (a) to (f) represent respectively the regions corresponding to the densities, n_0 (cm^{-3}) = 10^{30} , 10^{32} , 10^{33} , 5×10^{33} , 7×10^{33} and 10^{34} . The critical value at which the stable (unstable) regions are shifted to unstable (stable) ones is $n_0 = 7 \times 10^{33} \text{ cm}^{-3}$, and above which the stability region increases slightly with n_0 [see Fig. 1(f)]. The panels (a), (e) and (f) show that the regions of stability and instability are divided into two parts at both lower and higher densities.

we have seen from Fig. 1 that the waves tend to become stable at higher number densities. Thus, from this section we conclude that the number densities of ultra-relativistic ultra-cold electrons significantly modify the stability and instability regions in the $k\theta$ -plane as well as reduce the instability growth rate for the wave packets. In the next section, we will numerically investigate the dynamical evolution of the EWPs that undergo MI in the regimes discussed above. We will see that the situation is quite distinctive from the one-dimensional case in which an exact balance between the self-focusing and the dispersion can happen to form envelope solitons on stable wave propagation.

IV. 2D EVOLUTION OF THE NONLOCAL EQUATIONS

In the evolution of EWPs described by the Eqs. (10) and (11) we first obtain some analytic conditions for the wave collapse to occur within a finite time. Note that since the evolution equations are of the DS II-type, we can not derive the criteria that are sufficient to ensure collapse by the Virial theorem or else¹¹. However, we will present some conditions according to Berkshire and Gibbon²³. To that end, we first see that the integrals of motion are the wave action N and the Hamiltonian \mathcal{H} where

$$N = \iint |\phi|^2 d\xi d\eta, \quad (21)$$

$$\begin{aligned} \mathcal{H} = & \iint \left[P_1 \left| \frac{\partial \phi}{\partial \xi} \right|^2 + P_2 \left| \frac{\partial \phi}{\partial \eta} \right|^2 + \frac{1}{2} Q_1 |\phi|^4 \right] d\xi d\eta \\ & - \frac{Q_2}{2S} \iint \left[R \left(\frac{\partial^2 u}{\partial \xi^2} \right)^2 + \left(\frac{\partial^2 u}{\partial \xi \partial \eta} \right)^2 - 2S \frac{\partial^2 u}{\partial \xi^2} |\phi|^2 \right] d\xi d\eta, \end{aligned} \quad (22)$$

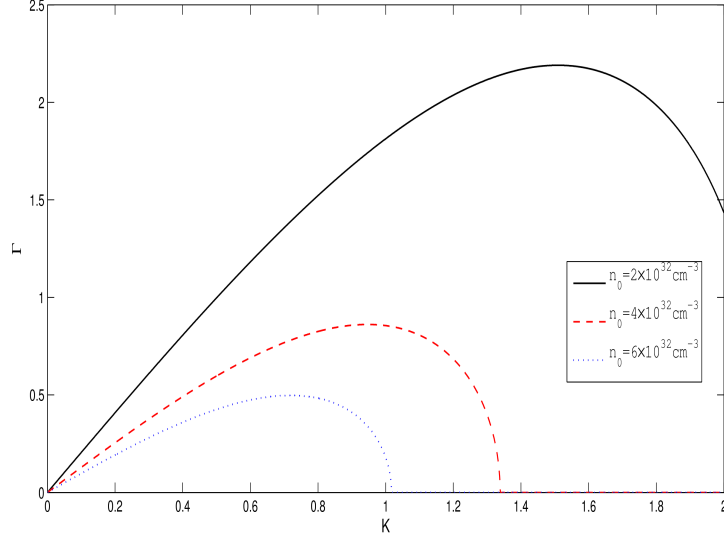


FIG. 2. (Color online) The modulational instability growth rate, Γ is shown with respect to the wave number of modulation, K for a fixed $k = 0.6$, $\zeta = 0.2$ and $\phi_0 = 0.1$. This shows that the higher the particle density, the lower is the growth rate, Γ with cut-offs at lower K .

where $\partial^2 u / \partial \xi^2 \equiv \psi$. If I is the moment of inertia of a localized wave form, then by the Virial theorem we have¹¹

$$\frac{d^2 I}{d\tau^2} \equiv \frac{d^2}{d\tau^2} \left[\iint \left(\frac{\xi^2}{P_1} + \frac{\eta^2}{P_2} \right) |\phi|^2 d\xi d\eta \right] = 8\mathcal{H}, \quad (23)$$

which upon integration gives

$$\frac{dI}{d\tau} = 8\mathcal{H}\tau + C_1, \quad (24)$$

or,

$$\langle I \rangle \equiv \frac{I}{N} = \frac{4\mathcal{H}}{N} \tau^2 + C_1 \tau + C_2. \quad (25)$$

The constants C_1 and C_2 are then given by

$$C_1 = \left\langle \frac{dI}{d\tau} \right\rangle \Big|_{\tau=0} = \frac{dI(0)}{d\tau}, \quad C_2 = \langle I \rangle \Big|_{\tau=0} = I(0). \quad (26)$$

Thus, according to Ref.²³, the conditions for the collapse to occur in a finite time are $\langle d^2 I / d\tau^2 \rangle \geq 0$, $C_1(0) < 0$, $C_2(0) > 0$ and that the angular momentum integral vanishes, $J(\tau=0) = 0$. Furthermore, if $C_1 = 0$, i.e. $dI/d\tau = 8\mathcal{H}\tau$, then if $\mathcal{H} > 0$ an initial waveform stretched along ξ -axis will evolve into a structure stretched along the η -axis for sufficiently large times. On the other hand, for $\mathcal{H} < 0$, the wave will be stretched along the ξ -axis. The detailed analysis is, however, beyond the scope of the present study, instead we will focus on the numerical evolution of the nonlocal NLSEs (10) and (11). We find that whatever may be the initial waveform (e.g., 2D Gaussian or close to the exact solution), the wave amplitude either decays due to dispersion or blows-up in a finite time due to nonlinearity, when the wave action to the initial condition is below or above the threshold. Furthermore, for a particular value of the power of the initial condition close to the exact solution, we find that the coherent structures, such as lumps, can also propagate almost without any change.

The behaviors of the solutions of the DS II equations are quite well-known and have been investigated by many authors in the context of water wave propagation with finite depth (see, e.g. Refs.^{20,21}). However, the solution is still not well understood or almost unknown in the context of plasma physics. To our knowledge, few authors have investigated the MI (see, e.g. Refs.¹⁷⁻¹⁹) of a plane wave packet as well as some analytic solution (see, e.g.¹⁹) of the DS II-like equations in plasmas.

In our numerical scheme we consider a space domain as $[-15, 15] \times [-15 \times 15]$ with 150 grid points in every direction and time step $\delta t = 10^{-3}$. The parameter values which satisfy the MI condition (18) are taken as $P_1 = -0.317$,

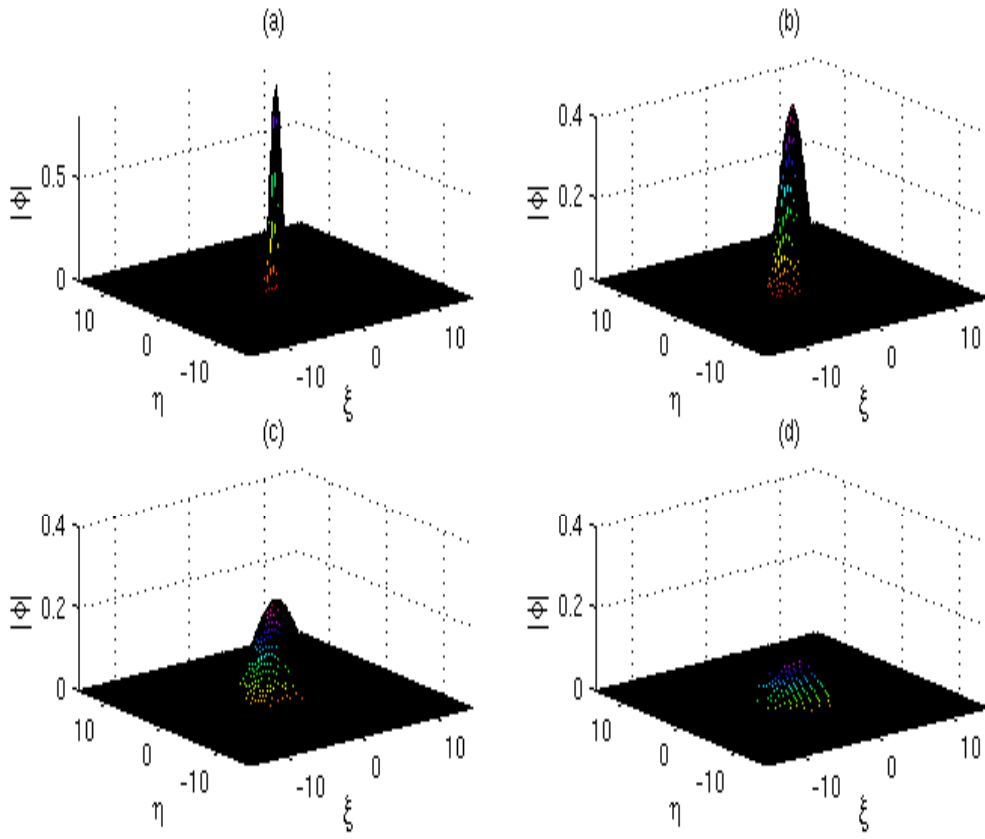


FIG. 3. (Color online) An initial Gaussian waveform decays in a finite time due to enhancement of the dispersion by the static field: (a) $\tau = 0$, (b) $\tau = 0.5$, (c) $\tau = 1$ and (d) $\tau = 2$.

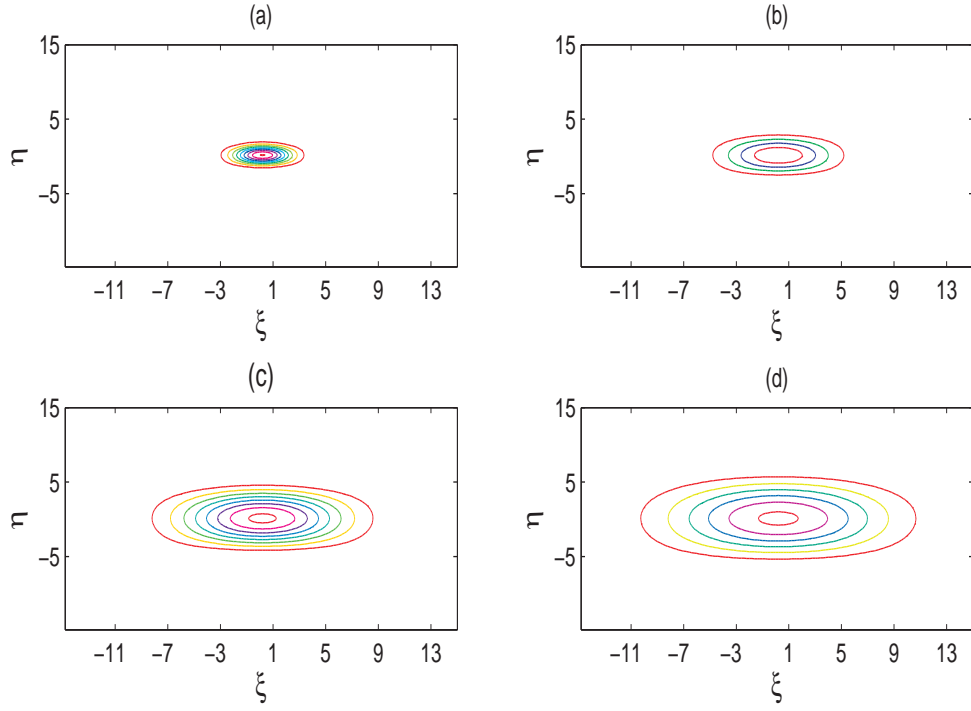


FIG. 4. (Color online) Contour plots of the profiles ($|\phi| = \text{const.}$) as in Fig. 3. The evolution is shown at (a) $\tau = 0.5$, (b) $\tau = 1$, (c) $\tau = 1.5$ and (d) $\tau = 2$.

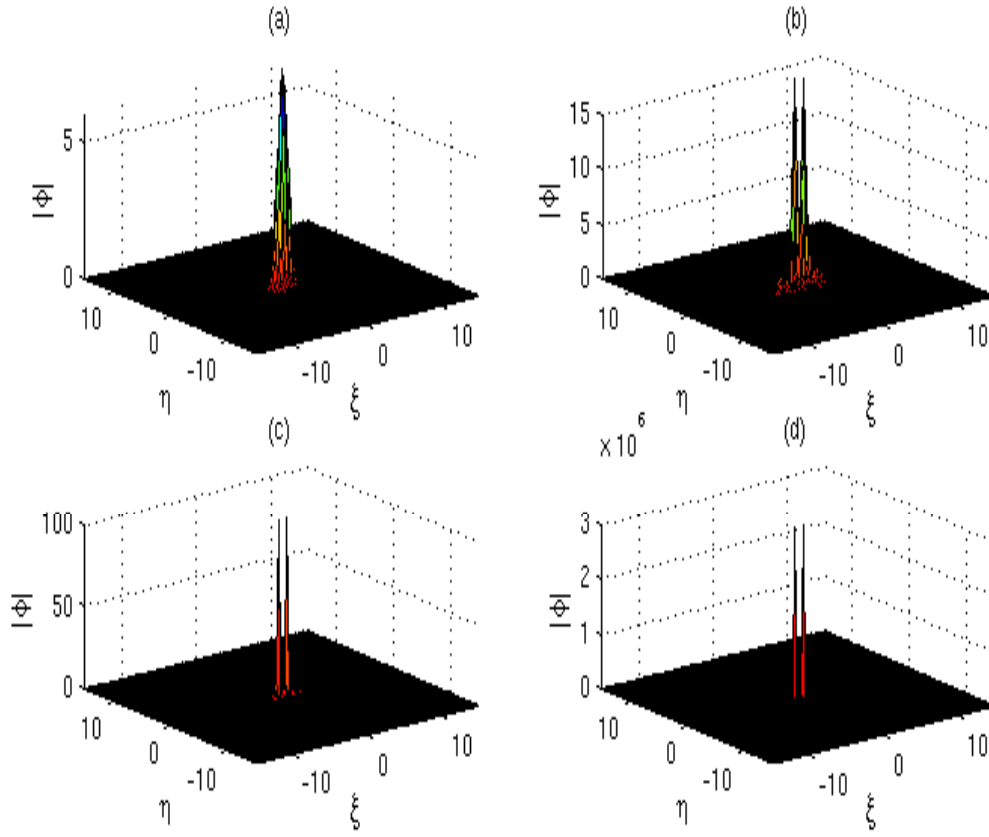


FIG. 5. (Color online) Double focusing solutions that blowup in a finite time. The initial wave form is Gaussian with amplitude $A = 8$. (a) Initial state: $\tau = 0$, (b) Intermediate state: $\tau = 0.35$, (c) Before blowup: $\tau = 0.356$ and (d) Shortly after blowup ($|\phi| \sim 10^6$): $\tau = 0.358$.

$P_2 = 0.9327$, $Q_1 = -0.1686$, $Q_2 = 0.2876$, $R = 0.3028$ and $S = -4.2086$ corresponding to $k = 0.4$ and $n_0 = 2 \times 10^{33} \text{ cm}^{-3}$, i.e. $\beta = 0.7985$. For the sake of simplicity, we consider a symmetric Gaussian profile as the initial condition, i.e. $\phi = A \exp(-\xi^2/|P_1| - \eta^2/P_2)$. We have tested our numerical results with other forms of initial conditions, namely $\phi = A/(1 + \xi^2/|P_1| + \eta^2/P_2) \exp(i\xi^2/P_1 + i\eta^2/P_2)$, which is a localized lump with algebraic decay and also $\phi = [A/(1 + \xi^2/|P_1| + \eta^2/P_2)] \exp(2i\eta/\sqrt{P_2})$, which approximates an exact solution of the DS II-like equations^{20,21}, but observe the similar qualitative features as presented here.

Figure 3 shows that for $A = 1$, the initial wave forms decay with time, and go to zero after a finite time. The corresponding contour plots are shown in Fig. 4, but for different times. In this case, the presence of the nonlocal static field enhances the dispersion of the profile and since the quadratic nonlinearities (nonlocal term) are known to be collapse free, the initial wave action N is not up to the mark for which the cubic nonlinearity dominates. However, since the effect of the static field ψ is to modify the nonlinearity originating from the zeroth harmonic modes (or slow modes), we expect that the coupling of the field ψ and the wave field amplitude ϕ can drastically affect the evolution dynamics for blowup. To elucidate it we have considered $A = 8$, which is above the critical power N of blowup for a Gaussian pulse whose evolution is described by the 2D NLSE. Thus, Figs. 5 and 6 (contour plot) show that the initial waveform is not preserved, and wave singularity is formed at two points within a finite interval of time leading to double focusing effect. The wave packet thus blows-up in shorter scales with higher amplitudes. In this case, the self-focusing effects dominate over the dispersion as required. Evolution of such wave collapse could be an effective mechanism for energy localization in URD dense plasmas.

A slightly different perspective is observed when considering the initial profile as $\phi = [A/(1 + \xi^2/|P_1| + \eta^2/P_2)] \exp(2i\eta/\sqrt{P_2})$, i.e. close to the exact solution of DS II equations^{20,21}. Figure 7 shows that for $A = 1$ the coherent structure propagates along the ξ -axis almost without any change (after a certain interval) of its amplitude. We can, however, precisely deal with such cases by choosing the initial condition as close to the exact solution of the NLSEs (10) and (11) as well as the proper wave action.

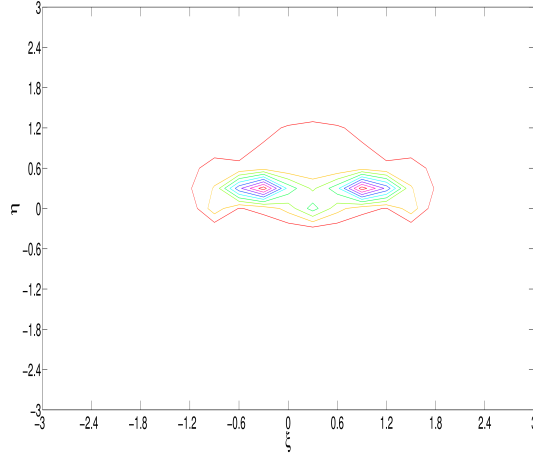


FIG. 6. (Color online) Contour plot ($|\phi| = \text{const.}$) of a double focusing solution that blows up (as in Fig. 5) is shown at an intermediate time $\tau = 0.35$.

V. CONCLUSION

We have investigated the multi-dimensional modulation of an electrostatic wave packet propagating in an ultra-relativistic ultra-cold degenerate dense plasma. The dynamics of such wave packets is described by a coupled set of nonlocal nonlinear Schrödinger-like equations that involve a nonlocal (quadratic) nonlinear term. The latter appears due to the static wave field originating from the mean motion (zeroth harmonic) in the plasma. The equations are then used to obtain the instability condition for the modulation of a plane wave packet. It is shown that the density dependent parameter β , which arises due to the ultra-relativistic pressure of degenerate electrons, shifts the stable (unstable) region at lower density ($n_0 \sim 10^{30} \text{ cm}^{-3}$) to unstable (stable) regions at comparatively higher densities ($n_0 \sim 10^{33} \text{ cm}^{-3}$), and also that the stable and unstable regions are completely separated in such regimes. The latter can be achievable, e.g. in the interior of massive white dwarfs and neutron stars⁵. Furthermore, the instability growth rate is obtained and found to be lowered at higher number densities with cut-offs at lower wave numbers of modulation. This implies that the role of β is analogous with that due to quantum dispersion associated with the Bohm de Broglie potential for the MI of wave envelopes in quantum plasmas^{7–10}.

We have also shown that the presence of the static field can drastically change the dynamical evolution of the wave packets quite distinctive from the one-dimensional wave packets or 2D NLSE. We found that when the initial condition is either a Gaussian pulse or a localized lump with algebraic decay, the wave amplitudes either disperse away to zero or blowup to infinity at singular points when the wave action (power) is below or above a threshold value. Such a blowup mechanism could be important for the energy localization in dense plasmas where the electrons are ultra-relativistically degenerate. On the contrary, when an initial waveform is close to the exact solution of the DS II-like equations^{20,21}, the coherent structure propagates without almost any change. However, confirmation of this behavior needs further detail numerical investigation.

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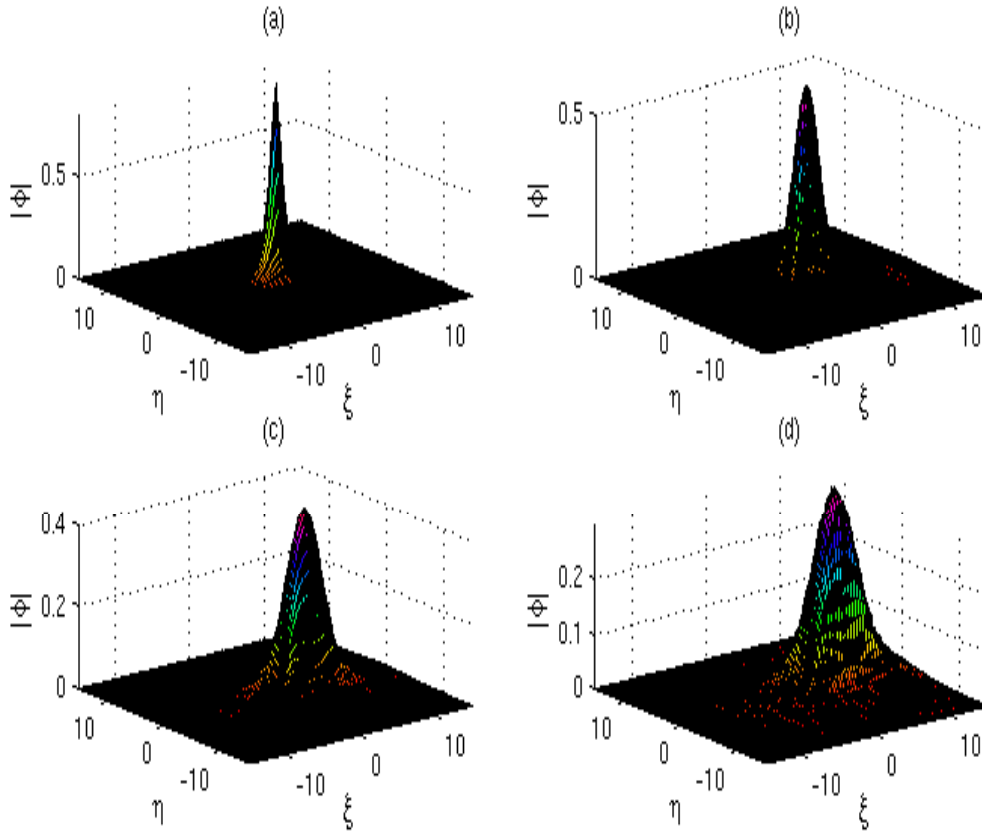


FIG. 7. (Color online) Propagating dromions (when the initial waveform is close to the exact solution of the DS-II equations) at different times: (a) $\tau = 0$, (b) $\tau = 0.5$, (c) $\tau = 1$ and (d) $\tau = 1.5$.

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